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# ASSOCIO-SYMMETRIC ALGEBRAS

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**Abstract.** Let $A$ be an algebra over a field $F$ satisfying $(x, x, x)=0$ with a function $g: A\times A\times A\to F$ such that $(xy)z=g(x, y, z)x(yz)$ for all $x, y, z$ in $A$. If $g(x_1, x_2, x_3)=g(x_{1\pi}, x_{2\pi}, x_{3\pi})$ for all $\pi$ in $S_3$ and all $x_1, x_2, x_3$ in $A$ then $A$ is called an associo-symmetric algebra. It is shown that a simple associo-symmetric algebra of degree $>2$ or degree $=1$ over a field of characteristic $\neq 2$ is associative. In addition a finite-dimensional semisimple algebra in this class has an identity and is a direct sum of simple algebras.

Throughout we shall let $A$ denote an associo-symmetric algebra over a field $F$ of characteristic $\neq 2$. In §1 we show that $A$ is power-associative and has a vector space decomposition $A=A_{11}+A_{10}+A_{01}+A_{00}$ relative to any idempotent $e$. In §2 the multiplicative properties of the submodules are studied and as a consequence of these one obtains Theorem 3.2 that if $A$ is simple and $e\neq 1$ is an idempotent then $A_{11}(e)$ and $A_{00}(e)$ are associative subalgebras. The decomposition of $A$ relative to several orthogonal idempotents, derived in §4, is used to obtain Theorem 4.2 that if $A$ is simple and has degree $>2$ then $A$ is associative. The main result of §5 is that if $A$ is finite dimensional and semisimple then $A$ has an identity and is a direct sum of simple algebras. Finally in §6 an argument is adopted from alternative rings to show that if $A$ is simple and of degree one then it is a field.

## 1. Preliminaries.

THEOREM 1.1. *If $A$ is an associo-symmetric algebra then $A$ is power-associative.*

**Proof.** We show that $x^a x^b = x^{a+b}$ for any $x$ in $A$ by induction on $k=a+b$. The result holds if $k=3$ by third power-associativity. Assume that the result holds for all $k<n$ and let $0<s\leqq n-1$. Then $x^{n-1}x=(x^{n-s-1}x^s)x=g(x^{n-s-1}, x^s, x)x^{n-s-1}(x^{s+1})$. On the other hand $x^{n-s}x^s=(x^{n-s-1}x)x^s=g(x^{n-s-1}, x, x^s)x^{n-s-1}x^{s+1}$. By associo-symmetry, however, $g(x^{n-s-1}, x^s, x)=g(x^{n-s-1}, x, x^s)$. Therefore $x^{n-1}x=x^{n-s}x^s$ and if we let $a=n-s$, $b=s$ then $x^a x^b=x^n=x^{a+b}$. Thus, $A$ is power-associative by finite induction.

It seems worthwhile to remark here that the assumption $(xy)z=g(x, y, z)x(yz)$ with $g: A\times A\times A\to F$ and $g(x_1, x_2, x_3)=g(x_{1\pi}, x_{2\pi}, x_{3\pi})$ is not in itself sufficient to

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guarantee power-associativity even in finite dimension, as the following example indicates. Let $A$ have basis $a, b, c, d, e, f$ over a field of characteristic $\neq 2$ with multiplication given by $ab=c$, $cd=e$, $bd=2f$, $af=e$, and all other products zero. Then $g(x, y, z)=\frac{1}{2}$ for all $x, y, z$ in $A$. However

$$(a+b+d)^2(a+b+d) = (c+2f)(a+b+d) = e$$

and

$$(a+b+d)(a+b+d)^2 = (a+b+d)(c+2f) = 2e.$$

Therefore $A$ is not power-associative.

LEMMA 1.1. *Let $e$ be an idempotent of an associo-symmetric algebra $A$ over a field of characteristic $\neq 2$. Then $(a, e, e)=(e, a, e)=(e, e, a)=0$ for all $a$ in $A$. (Here $(a, b, c)=(ab)c-a(bc)$.)*

**Proof.** Since $A$ is power-associative, $A$ has a vector space decomposition $A=A_1+A_{1/2}+A_0$ relative to $e$ where $A_i=\{a_i \mid ea_i+a_ie=2ia_i\}$ for $i=1, 0, \frac{1}{2}$ [1]. Since $A$ satisfies $(x, x, x)=0$ it can be shown (see [6, p. 137]) that

$$A_i = \{a_i \mid ea_i = a_ie = ia_i\} \quad \text{for } i = 1, 0.$$

Now let $a\neq 0$ be in $A$. From $(x, x, x)=0$ we have $(e, e, a)+(e, a, e)+(a, e, e)=0$. If we let $\alpha=g(e, e, a)=g(a, e, e)=g(e, a, e)$ then by the associo-symmetric identity, $(\alpha-1)e(ea)+(\alpha-1)e(ae)+(\alpha-1)ae=0$. If $\alpha=1$ the lemma follows. Otherwise $e(ea)+e(ae)+ae=0$. Let $a=a_1+a_{1/2}+a_0$. Then $ae=a_1+a_{1/2}e$, $e(ae)=a_1+e(a_{1/2}e)$, and $e(ea)=a_1+e(ea_{1/2})$. Thus we have $3a_1+a_{1/2}e+e(a_{1/2}e+ea_{1/2})=0$. But $a_{1/2}e+ea_{1/2}=a_{1/2}$. Therefore we get $3a_1+a_{1/2}=0$, $a_{1/2}=0$ and $a=a_1+a_0$. (In fact if characteristic $F\neq 3$ then $a=a_0$.) Thus $(e, a, e)=(a, e, e)=(e, e, a)=0$.

It is a well-known fact that in any algebra $A$, the results of Lemma 1.1 imply that $A$ has a Peirce decomposition $A=A_{11}+A_{10}+A_{01}+A_{00}$. Therefore we have

THEOREM 1.2. *If $A$ is an associo-symmetric algebra over a field of characteristic $\neq 2$ and if $e$ is an idempotent of $A$ then $A=A_{11}(e)+A_{10}(e)+A_{01}(e)+A_{00}(e)$ where $A_{ij}(e)=\{x_{ij} \mid ex_{ij}=ix_{ij}$ and $x_{ij}e=jx_{ij}\}$.*

It is clear that if $a=a_{11}+a_{10}+a_{01}+a_{00}$ then $a_{11}=eae$, $a_{10}=ea-eae$, $a_{01}=ae-eae$, and $a_{00}=a-ae-ea+eae$.

## 2. Multiplication of the modules.

LEMMA 2.1. $(A_{11}+A_{01})(A_{00}+A_{01})=0$.

**Proof.** Let $x\in A_{11}+A_{01}$, $y\in A_{00}+A_{01}$. Then $xy=(xe)y=g(x, e, y)x(ey)=0$.

LEMMA 2.2. $A_{11}A_{11}\subseteq A_{11}$, $A_{11}A_{10}\subseteq A_{10}$.

**Proof.** Let $x, y\in A_{11}$ and $g(e, x, y)=\alpha$. Then $xy=(ex)y=\alpha e(xy)$. If $\alpha=0$ then $xy=0\in A_{11}$. Suppose $\alpha\neq 0$ and $xy=a_{11}+a_{10}+a_{01}+a_{00}$. Then $xy=\alpha(a_{11}+a_{10})$ or $a_{11}+a_{10}+a_{01}+a_{00}=\alpha(a_{11}+a_{10})$. The vector space direct sum then forces $\alpha=1$

and $a_{01}=a_{00}=0$. Therefore $xy\in A_{11}+A_{10}$. On the other hand $g(x, y, e)=\alpha=1$, $(xy)e=xy$. Therefore $xy\in A_{11}+A_{01}$. Thus, $xy\in(A_{11}+A_{10})\cap(A_{11}+A_{01})=A_{11}$.

Now, let $x\in A_{11}$, $y\in A_{10}$ and $\alpha=g(x, e, y)$. Then $xy=(xe)y=\alpha x(ey)=\alpha xy$. If $\alpha=0$ then $xy=0\in A_{10}$. Otherwise $\alpha=1=g(e, x, y)$. Thus $xy=(ex)y=e(xy)$ and $xy\in A_{10}+A_{11}$. However, $(xy)e=x(ye)=0$. Therefore $xy\in A_{10}+A_{00}$. Thus, $xy\in A_{10}$.

LEMMA 2.3. $A_{01}A_{11}\subseteq A_{01}$, $A_{01}A_{10}\subseteq A_{00}$.

**Proof.** Let $x\in A_{01}$, $y\in A_{11}$, $\alpha=g(x, e, y)$. Then $xy=(xe)y=\alpha x(ey)=\alpha xy$. Therefore $\alpha=0$ or 1. If $\alpha=0$ then $xy=0\in A_{01}$. Otherwise $\alpha=1=g(e, x, y)=g(x, y, e)$. Therefore $0=(ex)y=e(xy)$ and $(xy)e=x(ye)=xy$. Thus, $xy\in(A_{01}+A_{00})\cap(A_{01}+A_{11})=A_{01}$.

Next let $x\in A_{01}$, $y\in A_{10}$. Clearly $xy\in A_{10}+A_{00}$ since $(xy)e=g(x, y, e)x(ye)=0$. Let $\alpha=g(x, e, y)=g(e, x, y)$. If $\alpha=0$ then $(xe)y=0$ and $xy=0\in A_{00}$. Otherwise $g(e, x, y)\neq 0$. Then $0=(ex)y=g(e, x, y)e(xy)$ and $e(xy)=0$. Thus $xy\in A_{01}+A_{00}$. But $xy\in A_{10}+A_{00}$. Therefore $xy\in A_{00}$.

LEMMA 2.4. $A_{10}(A_{10}+A_{11})=0$, $A_{10}A_{00}\subseteq A_{10}$, $A_{10}A_{01}\subseteq A_{11}$.

**Proof.** Let $x\in A_{10}$, $y\in A_{1i}$ for $i=0, 1$ and $\alpha=g(x, e, y)=g(e, x, y)$. Then $0=(xe)y=\alpha x(ey)=\alpha xy$. If $\alpha\neq 0$ then $xy=0$. Otherwise $\alpha=g(e, x, y)=0$. Then $xy=(ex)y=0e(xy)=0$. Therefore $xy=0$. Next let $x\in A_{10}$, $y\in A_{00}$. Then $xy=(ex)y=\alpha e(xy)$. As in the proof of Lemma 2.2 this forces $\alpha=0$ or 1. If $\alpha=0$ we are done. If $\alpha=1$ then $xy\in A_{10}+A_{11}$. But $(xy)e=0$. Therefore $xy\in A_{10}+A_{00}$. Hence, $xy\in A_{10}$.

Finally, let $x\in A_{10}$, $y\in A_{01}$. Then $xy=(ex)y=\alpha e(xy)$. Again, $\alpha=0$ or 1. If $\alpha=0$ then $xy=0\in A_{11}$. Otherwise $\alpha=1$, $(xy)e=x(ye)=xy$ and $xy=(ex)y=e(xy)$. Thus $xy\in A_{11}$.

LEMMA 2.5. $A_{00}(A_{10}+A_{11})=0$, $A_{00}A_{01}\subseteq A_{01}$, $A_{00}A_{00}\subseteq A_{00}+A_{10}$.

**Proof.** Let $x\in A_{00}$, $y\in A_{1i}$ for $i=0, 1$. If $\alpha=1$ then $0=(xe)y=x(ey)=xy$. Otherwise $\alpha\neq 1$. Linearization of third power-associativity gives $(x, e, y)+(x, y, e)+(y, x, e)+(y, e, x)+(e, x, y)+(e, y, x)=0$ or

$$(\alpha-1)[x(ey)+x(ye)+y(xe)+y(ex)+e(xy)+e(yx)] = 0.$$

Since $\alpha\neq 1$ and by the definition of the modules, we have

$$xy+i(xy)+e(xy)+e(yx) = 0. \tag{1}$$

If $i=1$ then $yx\in A_{11}A_{00}=0$ by Lemma 2.1. Therefore $-\frac{1}{2}e(xy)=xy$, which forces $xy=0$. If $i=0$ then reconsider $\alpha$. If $\alpha=g(x, e, y)\neq 0$ then $0=(xe)y=\alpha x(ey)=\alpha xy$. Therefore $xy=0$. Otherwise $\alpha=0=g(e, y, x)$ and $yx=(ey)x=0$. Therefore (1) reduces to $xy+e(xy)=0$. This again forces $xy=0$ to show that $A_{00}(A_{10}+A_{11})=0$.

Now let $x \in A_{00}$, $y \in A_{01}$. If $\alpha=1$ then $(xy)e=x(ye)=xy$ and $e(xy)=(ex)y=0$. Therefore $xy \in A_{01}$. If $\alpha \neq 1$ then by a linearization of third power-associativity as in Lemma 2.5 we have $x(ey)+x(ye)+y(xe)+y(ex)+e(xy)+e(yx)=0$ which reduces to $xy+e(xy)+e(yx)=0$ since $x \in A_{00}$, $y \in A_{01}$. Now $yx \in A_{01}A_{00}=0$ by Lemma 2.1. Therefore $xy+e(xy)=0$ which forces $xy=0 \in A_{01}$. Therefore $A_{00}A_{01} \subseteq A_{01}$. Finally, the last statement of Lemma 2.5 is immediate.

The results of Lemmas 2.1–2.5 give the following.

THEOREM 2.1. *If $e$ is an idempotent of an associo-symmetric algebra $A$ over a field of characteristic $\neq 2$ then the modules $A_{ij}(e)$ have the multiplicative relations*

(2) $A_{11}A_{11} \subseteq A_{11}$,

(3) $A_{00}A_{00} \subseteq A_{00}+A_{10}$,

(4) $A_{ij}A_{kl}=0$ *if* $j \neq k$,

(5) $A_{ij}A_{jl} \subseteq A_{il}$ *unless* $i=j=l=0$.

It should be noted that if $A$ has an identity 1 and $e \neq 1$ then (3) can be strengthened to $A_{00}(e)^2 \subseteq A_{00}(e)$. For $A_{00}(e)^2 = A_{11}(1-e)^2 \subseteq A_{11}(1-e) = A_{00}(e)$.

3. **Simple algebras.** In an associative algebra the set $B=A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}$ is an ideal. We prove the same result for associo-symmetric algebras and use it to characterize the simple algebras. According to convention "simple" means "simple but not nil".

LEMMA 3.1. $A_{11}(A_{10}A_{01}) \subseteq A_{10}A_{01}$.

**Proof.** Let $x \in A_{11}$, $y \in A_{10}$, $z \in A_{01}$. If $\alpha=g(x,y,z)=g(x\pi, y\pi, z\pi)=1$ then $x(yz)=(xy)z \in A_{10}A_{01}$ by (5). Otherwise the linearization of $(a,a,a)=0$ gives

$$x(yz)+x(zy)+y(xz)+y(zx)+z(xy)+z(yx) = 0. \tag{6}$$

Now $x(zy) \in A_{11}A_{00}=0$, $xz \in A_{11}A_{01}=0$, and $yx \in A_{10}A_{11}=0$ by (4). Therefore (6) reduces to $x(yz)+y(zx)+z(xy)=0$. But $x(yz) \in A_{11}$, $y(zx) \in A_{11}$, and $z(xy) \in A_{00}$. Therefore $z(xy)=0$ and (6) reduces to $x(yz)=-y(zx)$. But $y(zx) \in A_{10}(A_{01}A_{11}) \subseteq A_{10}A_{01}$ by (5). Therefore $x(yz) \in A_{10}A_{01}$.

LEMMA 3.2. $A_{00}(A_{01}A_{10}) \subseteq A_{01}A_{10}$.

**Proof.** Let $x \in A_{00}$, $y \in A_{01}$, $z \in A_{10}$. If $g(x,y,z)=1$ then $x(yz)=(xy)z \in A_{01}A_{10}$. Otherwise we have (6). But $x(zy) \in A_{00}A_{11}=0$, $xz \in A_{00}A_{10}=0$ and so $y(xz)=0$, and $z(yx) \in A_{10}(A_{01}A_{00})=0$. Therefore (6) reduces to $x(yz)+z(xy)+y(zx)=0$. But $x(yz) \in A_{00}(A_{01}A_{10}) \subseteq A_{00}^2 \subseteq A_{00}+A_{10}$, $y(zx) \in A_{01}A_{10} \subseteq A_{00}$, and $z(xy) \in A_{10}A_{01} \subseteq A_{11}$. Therefore $z(xy)=0$ and $x(yz)=-y(zx) \in A_{01}A_{10}$ to prove the lemma.

THEOREM 3.1. *In any associo-symmetric algebra $A$ with idempotent $e$, $B=A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}$ is an ideal of $A$.*

**Proof.** Since $A=\sum_{i,j=0,1} A_{ij}$ it is sufficient to show that $A_{ij}B+BA_{ij} \subseteq B$ for $i,j=0,1$. Now the multiplicative properties in Theorem 2.1 and associo-symmetry immediately imply that $BA_{ij}\subseteq B$ for $i, j=0, 1$. Similarly, Theorem 2.1 implies that $A_{10}B+A_{01}B\subseteq B$. Consider $A_{11}B=A_{11}(A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10})$. Now $A_{11}A_{01} =A_{11}(A_{01}A_{10})=0$, $A_{11}A_{10}\subseteq A_{10}\subseteq B$ by Theorem 2.1 and $A_{11}(A_{10}A_{01})\subseteq A_{10}A_{01}\subseteq B$ by Lemma 3.1. We similarly use Lemma 3.2 to show that $A_{00}B\subseteq B$. Thus $B$ is an ideal of $A$.

COROLLARY 1. *If $e\neq 1$ is an idempotent of a simple associo-symmetric algebra $A$ then $A_{11}(e)=A_{10}(e)A_{01}(e)$ and $A_{00}(e)=A_{01}(e)A_{10}(e)$.*

**Proof.** Since $B$ is an ideal either $B=A$ or $B=0$. If $B=0$ then $A_{10}=A_{01}=0$. Hence $A_{00}A_{00}\subseteq A_{00}$. Thus $A=A_{11}(e)\oplus A_{00}(e)$ and $A_{11}(e)$, $A_{00}(e)$ are ideals of $A$. Since $e\notin A_{00}(e)$, $A_{00}(e)\neq A$. Therefore $A_{00}(e)=0$ and $A=A_{11}(e)$. But this contradicts the assumption that $e\neq 1$. Therefore $B=A$, $A_{11}(e)=A_{10}(e)A_{01}(e)$ and $A_{00}(e) =A_{01}(e)A_{10}(e)$.

COROLLARY 2. *If $e$ is an idempotent of a simple associo-symmetric algebra $A$ then $A_{00}(e)^2\subseteq A_{00}(e)$.*

**Proof.** Let $x, y\in A_{00}(e)$. Then by Corollary 1, $x=x_{01}x_{10}$ and $xy=(x_{01}x_{10})y =g(x_{01}, x_{10}, y)x_{01}(x_{10}y)$. The right-hand side is clearly in $A_{00}(e)$. Therefore the result follows.

LEMMA 3.3. *If $e$ is an idempotent of an associo-symmetric algebra $A$ and $A_{ij} =A_{ij}(e)$ then*

(a) $(A_{10}, A_{01}, A_{11})=0$,

(b) $(A_{11}, A_{10}, A_{01})=0$,

(c) $(A_{01}, A_{11}, A_{10})=0$.

**Proof.** The linearization of fourth power-associativity, $(x, x, x^2)=0$, gives

$$\begin{aligned}(7)\quad &(x, y, zw+wz)+(z, y, xw+wx)+(w, y, xz+zx)+(y, x, zw+wz)\\ &+(z, x, wy+yw)+(w, x, yz+zy)+(z, w, xy+yx)\\ &+(x, w, yz+zy)+(y, w, xz+zx)+(w, z, xy+yx)\\ &+(x, z, yw+wy)+(y, z, wx+xw) = 0 \qquad [6, \text{p. } 129].\end{aligned}$$

Let $x\in A_{10}$, $y\in A_{01}$, $z\in A_{11}$, and $w=e$. Then $zw+wz=2z$, $xw+wx=x$, and $wy+yw=y$. Also, by Theorem 2.1, $xz=zy=0$. Therefore for these specializations (7) reduces to

$$\begin{aligned}(8)\quad &2(x, y, z)+(z, y, x)+(e, y, zx)+2(y, x, z)+(z, x, y)+(e, x, yz)+(z, e, xy)\\ &+(z, e, yx)+(x, e, yz)+(y, e, zx)+(e, z, xy)+(e, z, yx)\\ &+(x, z, y)+(y, z, x) = 0.\end{aligned}$$

Now $(z, y, x)=0$ since $A_{11}A_{0i}=0$, $(y, x, z)=0=(x, z, y)$ since $A_{i0}A_{1j}=0$. Also $ey=e[y(zx)]=0$, $xe=e(yz)=0$. Therefore $(e, yz, x)=(x, e, yz)=0$. Also $ex=x$ and $e[x(yz)]=x(yz)$ since $x(yz)\in A_{11}$. Therefore $(e, x, yz)=0$. Similarly $(z, e, xy) =(z, e, yx)=(y, e, zx)=(e, z, xy)=(e, z, yx)=0$. Therefore (8) reduces to

$$(9)\qquad 2(x, y, z)+(z, x, y)+(y, z, x) = 0.$$

But third power-associativity gives

$$(x, y, z)+(z, x, y)+(y, z, x) = 0. \tag{10}$$

Therefore $(x, y, z)=0$ proving that $(A_{10},A_{01},A_{11})=0$. In (10) again $(z, x, y)+(y, z, x)=0$. But $(z, x, y) \in A_{11}$ and $(y, z, x) \in A_{00}$. Therefore $(z, x, y)=(y, z, x)=0$. Hence $(A_{11},A_{10},A_{01})=(A_{01},A_{11},A_{10})=0$.

LEMMA 3.4. *Under the same hypothesis as the previous lemma,* $(A_{10}, A_{00}, A_{01})=(A_{11}, A_{10}, A_{00})=0$.

**Proof.** Let $x \in A_{10}$, $y \in A_{00}$, and $z \in A_{01}$. Then third power-associativity reduces to $(x, y, z)+(y, z, x)+(z, x, y)=0$. But $(x, y, z) \in A_{11}$, $(y, z, x)+(z, x, y) \in A_{00}$. Therefore $(x, y, z)=0$. Similarly if $x \in A_{11}$, $y \in A_{10}$, and $z \in A_{00}$ we immediately obtain $(x, y, z)=0$.

THEOREM 3.2. *Let $e \neq 1$ be an idempotent of a simple associo-symmetric algebra $A$ over a field of characteristic $\neq 2$. Then $A_{ii}(e)$ is associative for $i=0, 1$.*

**Proof.** Let $x, y, z \in A_{11}=A_{11}(e)$. By the corollary to Lemma 3.1, $y=y_{10}y_{01}$ for some $y_{ij} \in A_{ij}$. Then $(xy)z=[x(y_{10}y_{01})]z$. By (b) of the previous lemma $x(y_{10}y_{01})=(xy_{10})y_{01}$. Therefore $(xy)z=[(xy_{10})y_{01}]z$. Since $xy_{10} \in A_{10}$ and by (a), $[(xy_{10})y_{01}]z=(xy_{10})(y_{01}z)$. Therefore $(xy)z=(xy_{10})(y_{01}z)$. By (b) again and since $y_{01}z \in A_{01}$, $(xy_{10})(y_{01}z)=x[y_{10}(y_{01}z)]$. Finally, by (a), $y_{10}(y_{01}z)=(y_{10}y_{01})z$. Therefore $(xy)z=x[(y_{10}y_{01})z]=x(yz)$ and $A_{11}$ is associative. Since $A_{00}(e)=A_{11}(1-e)$, $A_{00}$ is also associative.

4. **Decomposition relative to several idempotents.** If $A$ is an alternative or Jordan algebra and $e_1, e_2, \ldots, e_t$ are orthogonal idempotents of $A$, then one has a vector space decomposition $A=\sum A_{ij}$ $(i, j=0, 1, \ldots, t)$ with $A_{ij}=\{x \mid e_kx=\delta_{ki}x \text{ and } xe_l=\delta_{jl}x\}$ with $\delta$ the Kronecker delta. We show that the same decomposition is obtained for associo-symmetric algebras.

LEMMA 4.1. *Let $e, e'$ be orthogonal idempotents of an associo-symmetric algebra $A$ with 1. Then $e(e'x)=(xe')e=0$ and $(e, x, e')=0$.*

**Proof.** Let $A=A_{11}+A_{10}+A_{01}+A_{00}$ be the decomposition relative to the idempotent $e$. Then since $e$ and $e'$ are orthogonal, $e' \in A_{00}$. Thus if $x \in A$ then $x=x_{11}+x_{10}+x_{01}+x_{00}$ and $e'x \in A_{00}(A_{11}+A_{10}+A_{01}+A_{00}) \subseteq A_{01}+A_{00}$. (Here we are using the stronger form of (3) in an algebra with 1; namely, $A_{00}^2 \subseteq A_{00}$.) Therefore $e(e'x)=0$. Similarly $xe' \in A_{10}+A_{00}$ and so $(xe')e=0$. Now from third power-associativity either $g(e, x, e')=1$ or $e(xe'+e'x)+e'(xe+ex)+x(ee'+e'e)=0$ which reduces to $e(xe')+e'(xe)=0$. But $e(xe') \in A_{10}$, $e'(xe) \in A_{01}$. Therefore $e(xe')=0$ and $(ex)e'=g(e, x, e')e(xe')=0$ and in this case also $(e, x, e')=0$.

In routine fashion the previous lemma gives

THEOREM 4.1. *Let $e_1, e_2, \ldots, e_t$ be orthogonal idempotents of an associo-symmetric algebra $A$ with 1. Then $A=\sum A_{ij}$ $(i, j=0, 1, \ldots, t)$ is a vector space decomposition of $A$ with $A_{ij}=\{x \mid e_kx=\delta_{ik}x \text{ and } xe_l=\delta_{jl}x\}$.*

LEMMA 4.2. *$A_{ij}A_{kl}=0$ if $j\neq k$ $(i, j, k, l=0, 1, 2, \ldots, t)$.*

**Proof.** Either $j\neq 0$ or $k\neq 0$. If $j\neq 0$ then $A_{ij}\subseteq A_{11}(e_j)+A_{01}(e_j)$. But $A_{kl}\subseteq A_{01}(e_j)+A_{00}(e_j)$. Therefore $A_{ij}A_{kl}=0$. Similarly if $k\neq 0$.

We now prove the following fundamental theorem on associo-symmetric algebras.

THEOREM 4.2. *Let $A$ be a simple associo-symmetric algebra over a field of characteristic $\neq 2$ and let $1=e_1+e_2+\cdots+e_t$ for pairwise orthogonal idempotents $e_i$. Then if $t>2$, $A$ is associative.*

**Proof.** We shall be considering the Peirce decomposition $A=\sum A_{ij}$ relative to $e_1, e_2, \ldots, e_t$. Let $e=e_1+e_i$. Then $A_{11}(e)=eAe=A_{11}+A_{1i}+A_{i1}+A_{ii}$ is associative by Theorem 3.2. Therefore $(A_{1i}, A_{i1}, A_{1i})=0$. But $A_{10}(e_1)=\sum_{j=2}^t A_{1j}$ and $A_{01}(e_1)=\sum_{j=2}^t a_{j1}$. Now let $a, c\in A_{10}(e_1)$ with $b\in A_{01}(e_1)$. Then $a=\sum_{j=2}^t a_{1j}$, $c=\sum_{j=2}^t c_{1j}$, and $b=\sum_{j=2}^t b_{j1}$. Then $(ab)c=\sum_{j,k,l=2}^t (a_{1j}b_{k1})c_{1l}$. By the previous remark if $j=k=l$ then $(a_{1j}b_{k1})c_{1l}=a_{ij}(b_{k1}c_{1l})$. If $j\neq k$ then $a_{1j}b_{k1}=0$ by Lemma 4.2. Therefore $(ab)c=\sum_{j,l=2;j\neq l}^t (a_{1j}b_{j1})c_{1l}+\sum_{j=2}^t a_{1j}(b_{j1}c_{1j})$. But $a_{1j}\in A_{01}(e_j)=A_{10}(1-e_j)$, $b_{j1}\in A_{10}(e_j)=A_{01}(1-e_j)$, and $c_{1l}\in A_{00}(e_j)=A_{11}(1-e_j)$. Since, by Lemma 3.3, $(A_{10}, A_{01}, A_{11})=0$ we have $(a_{1j}b_{j1})c_{1l}=a_{1j}(b_{j1}c_{1l})$. Therefore $(ab)c=\sum_{j,l=2}^t a_{1j}(b_{j1}c_{1l})$. On the other hand $a(bc)=\sum_{j,k,l=2}^t a_{1j}(b_{k1}c_{1l})$. But $b_{k1}c_{1l}\in A_{10}(e_k)A_{00}(e_k)\subseteq A_{10}(e_k)$ and $a_{1j}\in A_{00}(e_k)$. Therefore $a_{ij}(b_{k1}c_{1l})=0$ if $j\neq k$ and $a(bc)=\sum_{j,l=2}^t a_{1j}(b_{j1}c_{1l})$ also. Thus $(ab)c=a(bc)$. A similar argument shows that $(A_{01}(e_1), A_{10}(e_1), A_{01}(e_1))=0$. Thus we have

LEMMA 4.3. *$(A_{10}(e_1), A_{01}(e_1), A_{10}(e_1))=(A_{01}(e_1), A_{10}(e_1), A_{01}(e_1))=0$.*

Lemma 4.3 together with Lemmas 3.3 and 3.4 is sufficient to prove the associativity of $A$ by showing that all associators $(A_{ij}(e_1), A_{kl}(e_1), A_{rs}(e_1))=0$. We show this for several cases which indicate the method to be used in general. Clearly $(A_{ij}, A_{kl}, A_{rs})=0$ if $j\neq k$ or $l\neq r$. By Lemmas 3.3, 3.4 and Theorem 3.2, $(A_{11}, A_{10}, A_{01})=(A_{11}, A_{10}, A_{00})=(A_{11}, A_{11}, A_{11})=0$. We show that $(A_{11}, A_{11}, A_{10})=0$. ($A_{ij}$ indicates $A_{ij}(e_1)$.) Let $x, y\in A_{11}$, $z\in A_{10}$. Then by the corollary to Theorem 3.1 $y=y_{10}y_{01}\in A_{10}A_{01}=A_{11}$. Then $(xy)z=[x(y_{10}y_{01})]z$. Since $(A_{11}, A_{10}, A_{01})=0$, $(xy)z=[(xy_{10})y_{01}]z$. But $xy_{10}\in A_{10}$, $z\in A_{10}$ and, by Lemma 4.3, $(A_{10}, A_{01}, A_{10})=0$. Therefore $(xy)z=(xy_{10})(y_{01}z)=x[y_{10}(y_{01}z)]$ using $(A_{11}, A_{10}, A_{00})=0$. Finally $y_{10}(y_{01}z)=(y_{10}y_{01})z$ by Lemma 4.3. Therefore $(xy)z=x[(y_{10}y_{01})z]=x(yz)$. We have shown that all associators which have an element of $A_{11}$ in the first place are zero.

We now consider associators having an element of $A_{10}$ in the first entry. We know that $(A_{10}, A_{00}, A_{01})=(A_{10}, A_{01}, A_{11})=(A_{10}, A_{01}, A_{10})=0$ by Lemmas 3.3, 3.4, and 4.3. What remains is $(A_{10}, A_{00}, A_{00})$. Let $x\in A_{10}$, $y, z\in A_{00}$. Then since $A_{00}=A_{01}A_{10}$, $y=y_{01}y_{10}$ and $(xy)z=[x(y_{01}y_{10})]z$. By Lemma 4.3, $x(y_{01}y_{10})=(xy_{01})y_{10}$. Therefore $(xy)z=[(xy_{01})y_{10}]z$. But $(A_{11}, A_{10}, A_{00})=0$. Therefore $(xy)z=(xy_{01})(y_{10}z)$. Finally we get $(xy)z=x[y_{01}(y_{10}z)]$ from $(A_{10}, A_{01}, A_{10})=0$. But $y_{01}\in A_{10}(1-e)$, $y_{10}\in A_{01}(1-e)$ and $z\in A_{11}(1-e)$. Therefore $y_{01}(y_{10}z)=(y_{01}y_{10})z$

and $(xy)z=x[(y_{01}y_{10})z]=x(yz)$, the desired result. The same arguments are used to show that associators with elements of $A_{01}(e_1)$ or $A_{00}(e_1)$ in the first entry are zero. Therefore $A$ is associative.

5. **Semisimple algebras.** A power-associative algebra $A$ is called semisimple if its nilradical = maximal nil ideal is zero. If $A$ is a finite-dimensional nonnil algebra than a familiar argument (see [5, p. 39]) shows that $A$ has a principal idempotent $e$. Clearly $e$ is a principal idempotent of $A^+$. In [3] Kokoris has shown that $A_{1/2}(e)+A_0(e)\subseteq \text{Rad } A^+$ (cf. proof of Lemma 1.1 for notation). But $A_{10}(e)+A_{01}(e)=A_{1/2}(e)$ and $A_{00}(e)=A_0(e)$. Therefore $A_{10}+A_{01}+A_{00}\subseteq \text{Rad } A^+$. Let $x\in A_{10}$, $y\in A_{01}$. Then $2x\cdot y=xy+yx\in \text{Rad } A^+$. Since $yx\in A_{01}A_{10}\subseteq A_{00}\subseteq \text{Rad } A^+$ we conclude that $xy\in \text{Rad } A^+$. Therefore $A_{10}A_{01}\subseteq \text{Rad } A^+$. Thus the ideal $B=A_{10}(e)A_{01}(e)+A_{10}(e)+A_{01}(e)+A_{01}(e)A_{10}(e)$ is a nil ideal. If we assume that $A$ is semisimple then $B=0$. Therefore $A_{10}(e)=A_{01}(e)=0$ and $A=A_{11}(e)+A_{00}(e)$. Since $A_{10}=0$, $A_{00}$ is a subalgebra and the sum is a direct sum $A=A_{11}\oplus A_{00}$. Since $e$ is principal $A_{00}$ is nil. Therefore $A_{00}=0$ and $A=A_{11}(e)$. Therefore $e$ is an identity element of $A$. We have proved the first part of the following theorem.

THEOREM 5.1. *Let $A$ be a finite-dimensional semisimple associo-symmetric algebra. Then $A$ has an identity and is the direct sum of simple algebras.*

To complete the proof assume that $D$ is an ideal of $A$. Since $D$ is not nil it has principal idempotent $e$. Thus, as before, $D_{10}+D_{01}+D_{00}\subseteq \text{Rad } D^+$ and $D_{10}D_{01}+D_{10}+D_{01}+D_{01}D_{10}$ is a nil ideal of $D$. Note however that $D_{10}=A_{10}$, $D_{01}=A_{01}$ and $D_{11}=A_{11}$. For $D_{10}=D\cap A_{10}\subseteq A_{10}$. On the other hand if $x\in A_{10}$ then $x=ex\in D$. Thus $A_{10}\subseteq D$ and $A_{10}=D_{10}$. Similarly for the others. Therefore $B=A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}$ is a nil ideal of $A$. Since $A$ is semisimple $B=0$. Therefore $A=A_{11}\oplus A_{00}=D_{11}\oplus A_{00}$. Since any ideal of $A_{ii}$ is automatically an ideal of $A$, $A_{ii}$ $(i=1, 2)$ is semisimple. Therefore $A$ is a direct sum of semisimple algebras of smaller dimension and an easy induction completes the theorem.

6. We close with a short discussion of the degree one case. If $A$ is a finite-dimensional associo-symmetric algebra whose only idempotent is the identity 1 over an algebraically closed field $F$, then an argument of Albert's [2, p. 526] shows that every element $a\in A$ is of the form $a=\alpha 1+n$ with $\alpha\in F$ and $n$ a nilpotent element.

THEOREM 6.1. *A finite-dimensional, simple degree one algebra over a field of characteristic $\neq 2$ is a field.*

**Proof.** Assume that $A$ is simple, degree one over $F$. We may assume without loss of generality that $F$ is algebraically closed. Then every $a$ in $A$ is of the form $\alpha 1+n$ and since $A$ is power-associative, if $\alpha\neq 0$ then $a$ has an inverse in $A$. Let $N=\{n\in A \mid n$ nilpotent$\}$. We show that $N$ is a subalgebra, hence an ideal of $A$. By Albert [2] and Oemhke [4], $N$ is a subspace of $A$. Let $x, y\in N$ with $y^n=0$, $y^{n-1}\neq 0$. If $xy$ is not nilpotent then $(xy)^{-1}$ exists in $A$. Then $y^{n-1}=[(xy)^{-1}(xy)]y^{n-1}=g((xy)^{-1}, xy, y^{n-1}) \times (xy)^{-1}[(xy)y^{n-1}]=g((xy)^{-1}, xy, y^{n-1})g(x, y, y^{n-1})(xy)^{-1}[xy^n]=0$. Therefore $y^{n-1}=0$, a contradiction. Hence $xy \in N$ and $N$ is an ideal of $A$. Since $A$ is simple $N=0$ and $A=F1$.

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